# Researching the center[1]

Fanny Albert, Marie Hubert, Benoît Jadin[2]

## Dia 1

**Prelude**

Last year, within the context of their final thesis to become secondary mathematics school teachers, Mary and Fanny have studied the central values and experimented various activities in classrooms. They'll talk about these experiments and we'll broaden the discussion on that basis.

**Clarification**

« The term "average value" typically refers to several different measures of center: arithmetic mean, median, mode, midrange, and in this paper we also use it for precursors to these. »
ARTHUR BAKKER AND KOENO P. E. GRAVEMEIJERAN, HISTORICAL PHENOMENOLOGY OF MEAN AND MEDIAN, Springer 2006

## Dia 2

Many of our contemporaries think they master statistics, because they are constantly confronted with them (in particular in the media). But is there a real conceptual understanding of central values ? Do the citizens really understand the data supplied by the society ? This is what we will try to understand through situations relatively close to everyday life. To achieve that, we have designed a questionnaire including 7 multiple choice questions in which one or several proposals could be selected with free text comments fields.

We collected 186 questionnaires from people aged between 16 and 84 years old. Among them; teachers (secondary, primary and nursery school), students, pensioners, farmers, workers, joiners, as well as accountants, engineers, employees (of office, bank and administration), doctors, nurses, physiotherapists, independents, and many others.

To clarify our analysis, we tried to distribute all these people in 4 categories:
1) The mathematics teachers (of the lower and upper secondary level) and the primary school teachers (9);
2) The pupils of the fifth year of secondary school with a course program oriented towards mathematics (28);
3) Certain students of the Bachelor program (year 1, 2 or 3) in mathematics (34);
4) The neutral public , ordinary people (115).
In our presentation, we shall limit ourselves to the analysis of three situations.

## Dias 3 and 4

**Average number of ears : how do you react if you are told that more than 99 % of the population has more ears than the average of the number of ears? It seems strange, but it's true.**
1. The poll was made on a population with many handicapped persons who lost an ear.
2. If we take 5 persons including one with only one ear it's true, but if we take the same example with 10 000 people, it will not work.
3. It is impossible.

First of all, the term "average" is often used, but not always adequately, which leads to a misunderstanding of the concept. Indeed, many people only associate the average with a calculation.

This first situation was not simple for everybody. Indeed, due to the complex formulation of the sentence and its rather strange content, many were a little surprised and sometimes discouraged by this context. The majority of people having marked one of the proposals thought that it was impossible.

[1] We thank Audrey and Isaline Jadin who have reread the text and made many corrections.
[2] All three were students and teacher, in 2016, at Helmo (Haute école libre mosane) Liège and are members of the GEM group (groupe d'enseignement mathématique) from Louvain-la-Neuve.

| Propositions | neutral public | math teachers | secondary school | high school |
|---|---|---|---|---|
| 1 | 12 % | 33 % | 7 % | 12 % |
| 2 | 16 % | 0 % | 7 % | 18 % |
| 3 | 30 % | 0 % | 54 % | 26 % |
| correct answer | 20 % | 67 % | 18 % | 38 % |

Tab 3

As a result, only 25% of respondents found the trick of this situation, which we could summarize in the following way :

It is true because only some people should have a single ear for the average to be lower than 2 ears. In this case, 99 % of people have more ears than the average of the number of ears.

We are also sharing this extract rather original, but completely correct from a logical point of view, although, as pointed out by one of the person who had been asked, there are many more extreme values downward (that is possessing 1 or 0 ear) than upward (that is by possessing 3 or more).

So logical... If an imbecile cut itself the ear, the average will be smaller than 2. Except if one other imbecile is born with 3 ears. And if there is the same number of imbeciles in both camps, the average stays 2.

Finally, some did not interpret the situation as we had thought, by not taking only into account the term "ear" as the part of the body. Indeed, they considered the musical ear or the fact to be a good listener, but also the internal and external ear as well as the atria of the heart. Thus the question was poorly understood in spite of the simple vocabulary used.

**Dias 5 and 6**

**Inhabitants per square kilometer: in Russia, on 1st January 2015, 146 267 288 inhabitants were considered as living there. Nevertheless, the number of inhabitants per km² is 8,4. By contrast, in Japan, where 127 103 388 inhabitants were counted in 2014, we observe a population density of 349 inhabitants per km². How to explain it ? What can be concluded ?**

1. Every Russian has 8,4 km² for himself.
2. There are many uninhabitable zones in Russia.
3. In Japan, we count 349 Japanese on each km² .
4. They miscounted the number of inhabitants.
5. The differences in the number of inhabitants per km² were too important for the population density to be accurately calculated.

This problem was rather well solved as 68,8 % of people marked the second solution which explained the situation by the important quantity of uninhabitable areas in Russia. Naturally, there are also such zones in Japan, but in smaller quantity. It has to be noticed that many were keen to complete their answer by indicating that the Russian territory was much larger than the Japan one.

| Propositions | neutral public | math teachers | secondary school | high school |
|---|---|---|---|---|
| **1** | 4 % | 11 % | 7 % | 0 % |
| **2** | 68 % | 78 % | 71 % | 67 % |
| **3** | 20 % | 11 % | 46 % | 26 % |
| **4** | 1 % | 0 % | 0 % | 3 % |
| **5** | 1 % | 0 % | 0 % | 0 % |
| **correct answer** | 36 % | 22 % | 29 % | 24 % |

Tab 5

Here is an explanation extracted from a questionnaire :

> Russia has a larger surface area, but the population is concentrated in certain places. There are thus square kilometers where there is 0 population, which results in a drop of the average.

A large part of the individuals also marked the proposal 3 "In Japan, we count 349 Japanese on every km$^2$.". Actually, this solution is false because if the average number of inhabitants per km² was 349, it would not mean in any way that every km² counts exactly 349. It is possible to find much more inhabitants on one given km$^2$ than on another one.

**Dias 7 and 8**

**Which score: in the United States, at the end of the secondary school, all the pupils have to take a general common examination called SAT (Test of School capacities). The average score obtained at this test was calculated based on a large number of schools in the country. This average score is 400. We randomly selected 5 pupils and recorded their results. The scores of the first four pupils are the following ones: 380, 420, 600 and 400. What is the most likely score for the fifth pupil?**

1. 200
2. 400
3. 410
4. All the scores are possible.
5. It is not possible to answer the question.

For this situation, the opinions are divergent. The answers are divided between the score 200 and the proposal "All the scores are possible", with however a slight preference for the latter. If we calculated the score which, added to the four others that were given and whose the sum is divided by 5, gave an average of 400, we found actually 200.

| Propositions | neutral public | math teachers | secondary school | high school |
|---|---|---|---|---|
| **1** | 35 % | 22 % | 25 % | 41 % |
| **2** | 3 % | 0 % | 4 % | 3 % |
| **3** | 1 % | 0 % | 4 % | 0 % |
| **4** | 57 % | 78 % | 57 % | 59 % |
| **5** | 19 % | 11 % | 32 % | 12 % |
| **correct answer** | 7 % | 22 % | 0 % | 3 % |

Tab. 7

However, this calculation cannot be realized without taking into account the sample ... (Very few people pinpointed this problem). Indeed:

| |
|---|
| We could think of a score of 200, but the sample is too small to have a strong probability that the average of the sample approaches the average of the population. |

We could thus expect all the scores, even if, the average being the only datum of this problem, the score 400 could have been proposed as it seems realistic, possible. However, it is not the most likely score (score which has the highest frequency namely the mode on which we do not have any information).

Furthermore, the number of possible results being very high, the probability that a given score goes out is very low (even nil). Besides, we do not know anything about the frequency distribution and thus about the representativeness of the average. It is therefore difficult to give one single answer.

At the end of these three analyses, we noticed surprisingly similar results in spite of a very diversified public. If all seem to be equal in the face of the questions asked, it seems that there might be no beneficial effect of the classic statistical education when dealing with this type of problems.

**Dia 9**

Following the analysis of the questionnaire, we realize that, whatever their age, numerous individuals are poorly equipped from a statistical point of view. Particularly with regard to the central values which do not seem to be mastered in depth. It is the reason why we consulted the programs of the beginning of the secondary school, as well as certain textbooks, and we collected some testimonies concerning the practice used within the classrooms.

**Dia 10**

In general, the objectives of the programs are close to each other : calculate the average and the mode (the median being often postponed to a later stage in the program) and compare populations or samples. We were surprised by the lack of demand for analyses arising from statistical studies.

**Dia 11**

Very few textbooks devote time to the learning of central values (uses, advantages and limitations). If the mean and the mode are sometimes explained, the median is rarely clarified. Furthermore, the mean is generally seen only as the result of a calculation and asked in a very abstract way, without any meaning or context.

Then, almost all of these books present the data only in the form of tables already indicating absolute and relative frequencies, cumulated frequencies, etc. Yet, the big wealth of statistics is actually to handle a raw series of data, to order them, to classify them, in order to analyze them, to compare them, to interpret them and to finally draw conclusions based on the previous steps. The research and analysis work is thus simplified and a part of the interest of statistics is lost.

Furthermore, most of the exercises are inserted a little bit artificially into poor contexts. Instead of proposing real situations involving statistics and why not, with a citizen vocation, the problems are disguised, what causes a lost of meaning from an educational point of view.

Finally, during the construction of graphs, the pupils are often required to realize one type of graph (whose shape, axis for the Cartesian, circle for circular, is sometimes drawn, what does not let the learners think about the scale to be used, for example), while it would be more instructive and more sensible to let the pupils choose the adequate graph and to make them justify their choice.

**Dia 12**

What is really done within classrooms ? What is the time really dedicated to data processing, to statistics and to probabilities ? It turns out that the chapters about data processing are sometimes (maybe often) placed at the end of the year, or even deleted due to the lack of time if not given to the pupils in autonomous work.

**Dia 13**

How to favor an in-depth learning of the central values in statistics for public of various ages and levels ? Firstly, let's make a brief review of certain senses and uses of the central values.
This part owes much to Arthur BAKKER (from the Freudenthal Institute of Utrecht) and Claudine SCHWARTZ (from the University of Grenoble).

**Dias 14 and 15**

In general, the central values allow to :
- Describe a population (in the statistical sense) with regard to a given variable. Example: "it is necessary to commute one hour to go from Liège to Brussels."
- Compare various populations. Example: international tests like PISA.
- Be located within a population. Example: a result obtained in a test will be compared with the average of the class.
- Represent a population. Example: "a grown-up giraffe weighs 830 kg, has a size of 4,6 m and a 60 kph speed." (Claudine SCHWARTZ) Do all the giraffes weigh 830 kg and measure 4,6 m ? Do they permanently move at 60 kph ? Is it their top speed ? How to interpret these results ?
- Eliminate fluctuations :
" We have to, above all, lose sight of the man taken in isolation, and consider him only a fraction of the species. By stripping him of his individuality, we shall eliminate all which is only accidental; and the individual peculiarities which have only few or no action on the mass will fade of themselves, and will allow to seize the general results."
(A. Quetelet, Sur l'homme et le développement de ses facultés ou Essai de physique sociale, Tome premier, Bachelier imprimeur-libraire, Paris, 1835.. Downloaded on the site http://gallica.bnf.fr)

**Dias 16 to 18**

Let's now have a closer look at the most known central value : the mean. It is generally used for :
- Reduce the errors, for example in measurement problems to approach the real value of a size.
- Estimate the size of an important population.
Examples :

- Former use : estimation of the number of leaves on the branch of a tree. To achieve it, a man has selected a single twig (that we do not imagine chosen at random) and has multiplied the number of present leaves on this one by the number of twigs on the branch. To choose the twig, the man has certainly opted for the one who appeared to him the most representative of all the twigs.
- Modern use : estimation of the number of participants during a demonstration through the selection of a cell representative of the people distribution within a grid system.

Notice that these first two uses are also valid for the median.
- Distribute evenly : representation of the size that would take each element of a series of data if they were all identical. In the example below, the average represents the number of CDs that each person would have if everybody was possessing the same thing. In a context of salaries, the average is the salary that everybody would have if we were all earning the same amount.

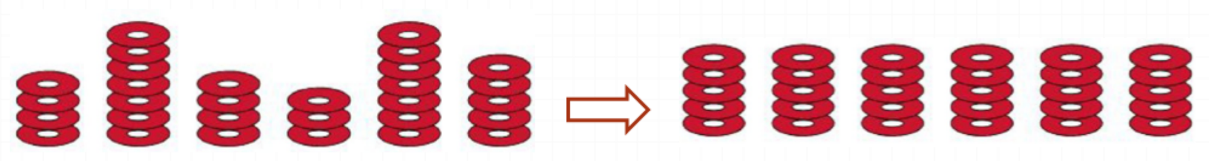


Fig. 18

Remark: to develop a conceptual understanding of the average, it is necessary to be conscious of two main points:

- Sense of the sum of the values : there are facilitating contexts to give a sense to the calculation of the average requiring this sum. For example, in a context of salaries, the latter represents the total payroll which has a sense, contrary to the sum of the temperatures of July which does not represent anything.
- Attention on the nonsense : there are different ways of interpreting the average according to the usage that is made. In principle, the calculation is identical, but the approach and the interpretation are totally different. For example, it does not make any sense to wonder what would be the size of each pupils of a classroom if all of them had the same size (contrarily to the example on salaries). Therefore, the use of "fair  sharing", in this context does not make any sense. . On the other hand, using this mean to characterize the average pupil has a sense, because each of them can be located in the classroom and we could also compare them with each other.

**Dias 19 to 21**

Starting from a graphical representation of frequencies, if we consider the *x* axis as an horizontal bar and every vertical bar as a weight exercised on this horizontal bar in a given point, the mean corresponds to the fulcrum which insures the balance of the horizontal bar. This one is a function of weights and of lever arms.

Let's consider a statistical series including two values with the same frequency (figure 19). The mean or pivotal point is just in the middle between both values. If the frequencies are not any more the same and one equals three times the other, the  lever arms have to be in the same proportion and are respectively equal to ¾ and ¼ of the distance between both values.

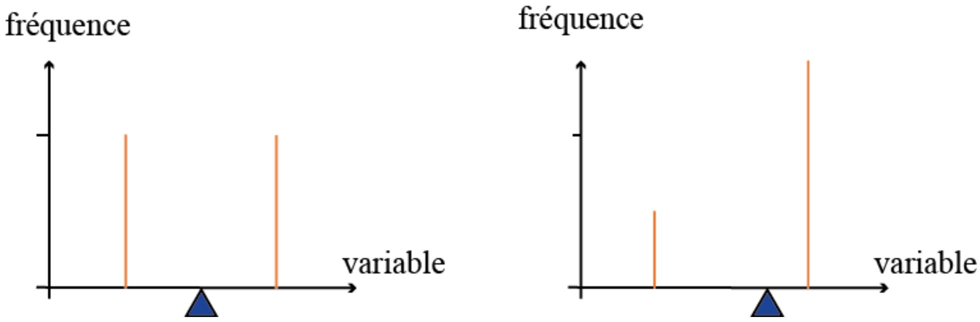



Fig. 19

If we consider a statistical series of several values (figure 21)… When the distribution is symmetric, the pivotal point is in the centre of the distribution. If the values remain the same but the weights change, this pulls the mean where the weights become more important. If the weights remain the same but the values change, the mean follows the values movement.

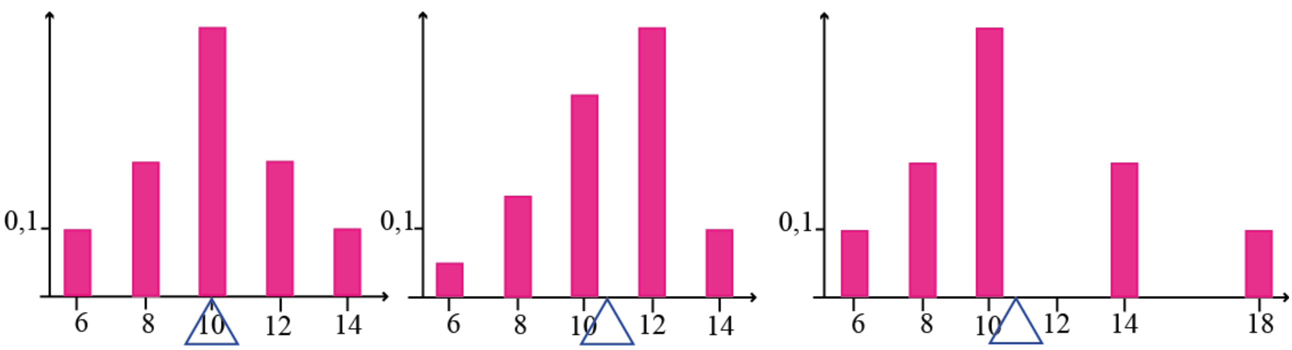



Fig. 21

**Dia 22**

We can say that the average has to be " the closest possible to all the numbers of the series ". What means the closest possible?

For a statistical series $(a_1, a_2, ..., a_n)$, we can define the dispersion of the squared deviations around any number $x$ as being

$$\varepsilon(x) = (a_1 - x)^2 + (a_2 - x)^2 + ... + (a_n - x)^2 \quad .$$

What value of $x$ minimizes this function ? By squaring and sorting the terms by degree, we obtain:

$$\begin{aligned}\varepsilon(x) &= (a_1 - x)^2 + (a_2 - x)^2 + ... + (a_n - x)^2 \\ &= nx^2 + (a_1{}^2 + a_2{}^2 + ... + a_n{}^2) - 2(a_1 + a_2 + ... + a_n)x\end{aligned}$$

$\varepsilon$ is a function of the second degree, the minimum of which is

$$\frac{2(a_1 + a_2 + ... + a_n)}{2n} \quad .$$

This value is nothing else than the mean of the series of values. We can therefore conclude that the central value, from the point of view of squared deviations, is the mean.

**Dia 23**

The midrange is a little-known central value, which corresponds to the mean of the smallest and the highest value of a dataset. Its main advantages are its ease and speed of calculation, and its intuitive side.

Historically, the first use of the midrange consisted in estimating the number of crew members of a fleet. Indeed, the purpose was to first identify the two boats containing the fewer and the more sailors and then to make the average of these two extremes values.

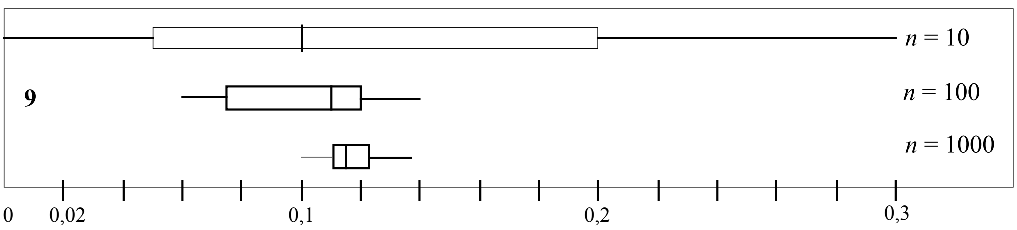
**Dias 24 and 25**

The median divides the total population in two equal parts and allows for a rapid perception of this population (or of the sample). The boxplot ("boîte à moustache" in french) provides a richer source of information, which still remains readily available.

Let's take an example to illustrate : The prince of Tuscany once asked Galilee: why when rolling three dices, do we obtain more often a total of 10 rather than a total of 9, although these totals are both achieved in six different ways ?

We simulated the experiment with a spreadsheet. It is faster than carrying out the experiment in reality. We first considered 20 series of 10 rolls . Then, 20 series of 100 rolls. And finally, 20 series of 1000 rolls. The figure 25 shows the results through a boxplot.
Let us start by considering the " release of 9 " :
- 25 % of frequencies are between 0 and 0,05.
- 25 % of the frequencies are between 0,05 and 0,1
- 25 % of the frequencies are between 0,1 and 0,2
- 25 % of the frequencies are between 0,2 and 0,3

When the sample size increases, the values tighten around the center (the median in this case), the sampling fluctuation decreases, the frequencies stabilize. We can compare medians in both cases and conclude that, generally, 10 is more frequent than 9.



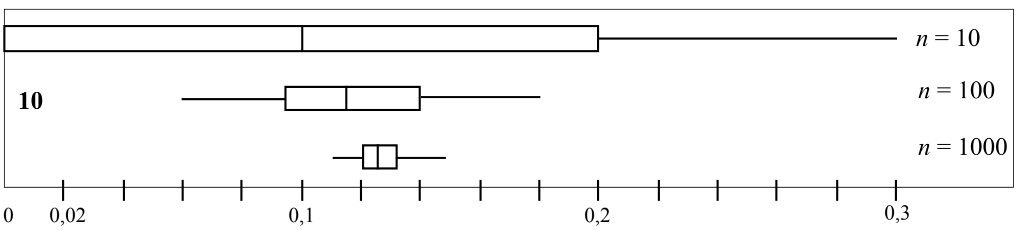




Fig. 25

**Dia 26 and 27**

John Graunt (or another one: Petty ?) established mortality tables of the London population at the beginning of the seventeenth century.

(Natural & Political Observations Upon the Bills of Mortality… of the City of London de 1662).

August 22nd, 1669, Louis Huygens (in Holland) raises the question to his brother Christian (in Paris) : « up to what age has a child to naturally live as soon as he is conceived ? »

(Œuvres complètes de C. Huygens, Tome 6 in Gallica *http://gallica.bnf.fr/Catalogue/noticesInd/FRBNF38949978.htm)*

Louis's calculation corresponds to the average lifetime, what we call "life expectancy" in a modern language. Whereas Christian's solution, starting from an robustness argument, bases on the median. At a given age, with a given number of survivors, he considers the time period over which the population has dropped by half.

**Dias 28 and 29**

On a bar chart, the median divides the total height of the bars into two equal parts. The figure 28 shows distributions of scores (on a total of 20).

The figure 29, on the other hand, is a histogram of the distribution of Belgian household incomes[3] in 2013. The median divides the coloured surface area into two equal parts.

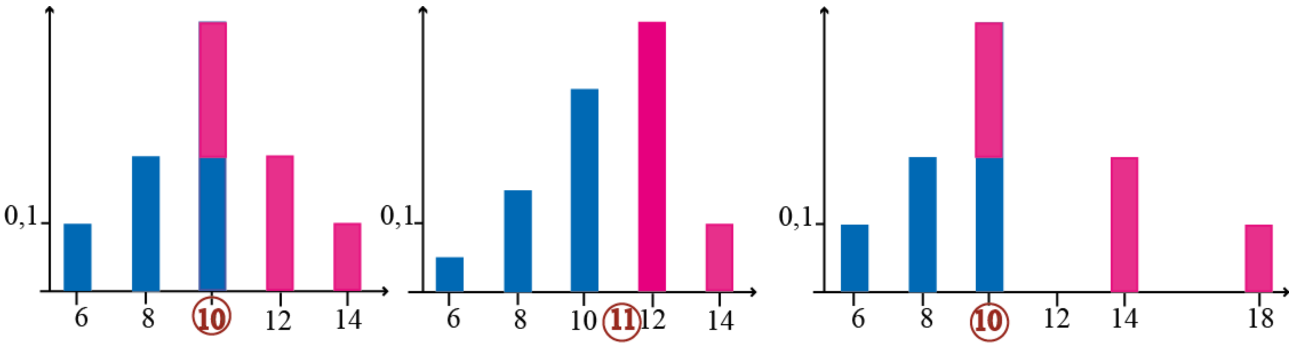



Fig. 28

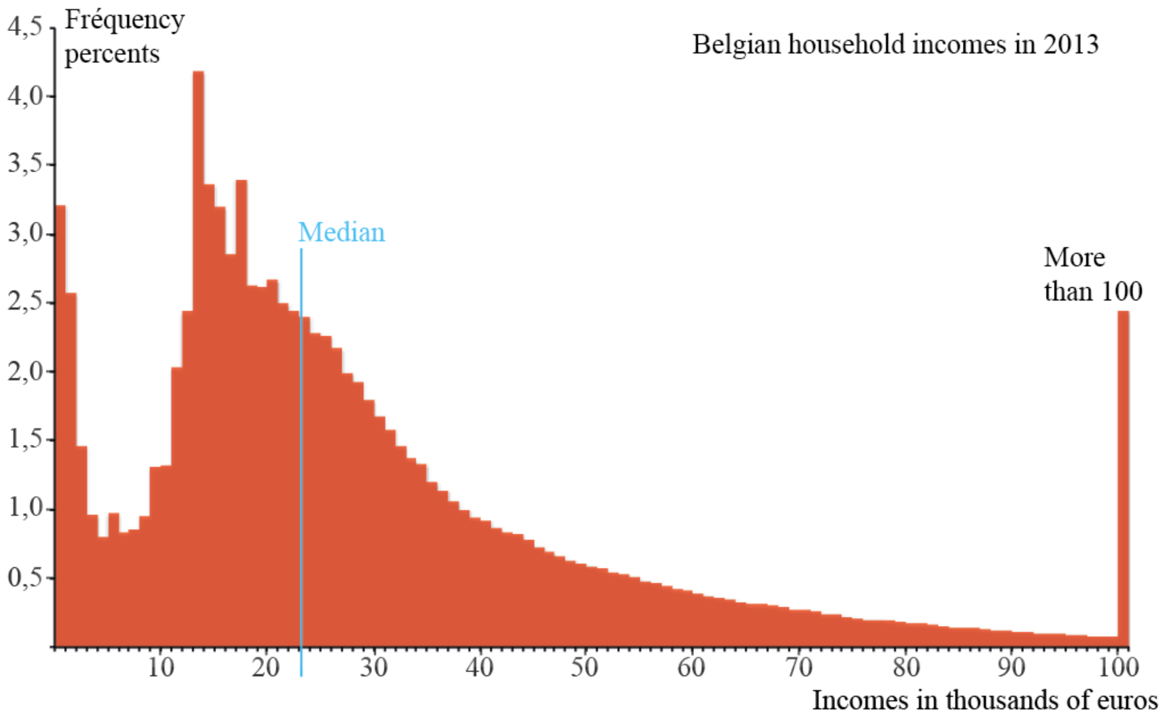



Fig. 29

[3] The data is downloaded from the website of the belgian government : http://economie.fgov.be/fr/statistiques/chiffres/travailvie/fisc/

**Dias 30 to 32**

We can say that the average has to be " the closest possible to all the numbers of the series ". What does mean the closest possible ?

For a statistical series, we are interested in the “absolute deviation” function for any value of *x* from the other values of the statistical series. It’s a linear piecewise function. What is the value of *x* that allows to minimize this function ?

$$\varepsilon(x) = |x - a_1| + |x - a_2| + ... + |x - a_n|$$

Let us take an example of educational outcomes on a total of 20 (table 31)

| $x$ | $f$ | $xf$ | $(x-\bar{x})$ | $(x-\bar{x})f$ | $\lvert x-\bar{x}\rvert f$ | $\lvert x-m\rvert f$ | $(x-\bar{x})^2 f$ | $(x-m)^2 f$ |
|---|---|---|---|---|---|---|---|---|
| 6 | 0,1 | 0,6 | -4,8 | -0,48 | 0,48 | 0,4 | 2,304 | 1,6 |
| 8 | 0,2 | 1,6 | -2,8 | -0,56 | 0,56 | 0,4 | 1,568 | 0,8 |
| 10 | 0,4 | 4 | -0,8 | -0,32 | 0,32 | 0 | 0,256 | 0 |
| 14 | 0,2 | 2,8 | 3,2 | 0,64 | 0,64 | 0,8 | 2,048 | 3,2 |
| 18 | 0,1 | 1,8 | 7,2 | 0,72 | 0,72 | 0,8 | 5,184 | 6,4 |
| | | 10,8 Mean | | 0 Average Deviation around the mean | 2,72 Average Absolute Deviation around the mean | 2,4 Average Absolute Deviation around the median | 11,36 Average Squared Deviation around the mean | 12 Average Squared Deviation around the median |

Tab. 31

We can see that the average squared deviation of the values of the series is smaller when considering the mean than when taking the median as the reference. If we look at the average absolute deviation, however, the opposite is true, the median is closer to the values than the mean.

Back to the function

$$\varepsilon(x) = |x - a_1| + |x - a_2| + ... + |x - a_n|$$

If $x \in [a_r, a_{r+1}]\,(1 \le r \le n-1)$, then

$$\begin{aligned}\varepsilon(x) &= (x-a_1) + ... + (x-a_r) + (a_{r+1} - x) + ... + (a_n - x)\\ &= (r-(n-r))x + (-a_1 - ... - a_r + a_{r+1} + ... + a_n)\\ &= (2r-n)x + (-a_1 - ... - a_r + a_{r+1} + ... + a_n)\end{aligned}$$

If $r < \frac{n}{2}$ , the slope of the line is negative and $\varepsilon$ is decreasing.

If $r > \frac{n}{2}$ , the slope of the line is positive and $\varepsilon$ is increasing.

If *n* is odd, $\varepsilon$ reaches its minimum in $r = n/2$ .

If *n* is even, $\varepsilon$ is constant and minimal on the interval $\left[\frac{n}{2}, \frac{n}{2}+1\right]$. We then choose the middle of the latter as median.

We can thus conclude that, when considering the absolute deviation, the central value best adjusted to the values of the series is the median.

**Dia 33**

The last central value that we would like to discuss is the mode, which represents the most frequent value, the one who returns most of the time. It is often used in elections contexts, symbolizing the choice of the majority (presidential elections in France, in the United States, etc.). Just like the average, the mode allows for estimation of large numbers. Indeed, in the famous

Athenian time, when people wanted to calculate the height of an enemy wall, it was asked to diverse persons to count the number of bricks. The height of the latter (including joints) being known, each person counted the number of bricks to obtain an estimation of the total height. Very logically, everybody did not obtain the same result. To arbitrate between the different results, the most frequent value was taken, namely the mode.

Let us note that in the XVIth century, in French, the term "mode" was only feminine. Only later, the mode was distinguished from the fashion. Yet, the latter have the same meaning, the fashion translating what the majority wears, what is trendy for the moment. Furthermore, this peculiarity represents a good mnemonic for the pupils. Finally, it is important to stress that the more modes there are, the less sense it makes. Actually, in a context of elections, when we obtain an equality, we repeat the process until a unique mode is obtained.

**Dia 34**

On a bar chart of frequencies, the mode represents a maximum (and a center in symmetric cases).

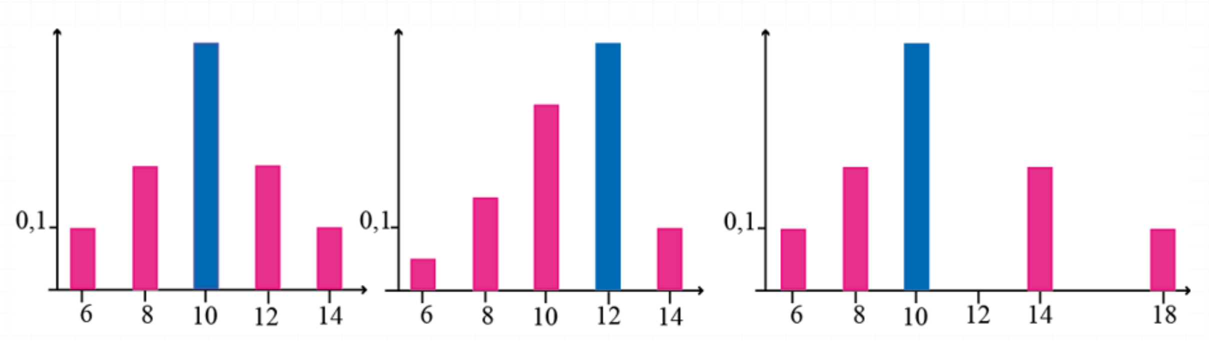



Fig. 34

**Dia 35**

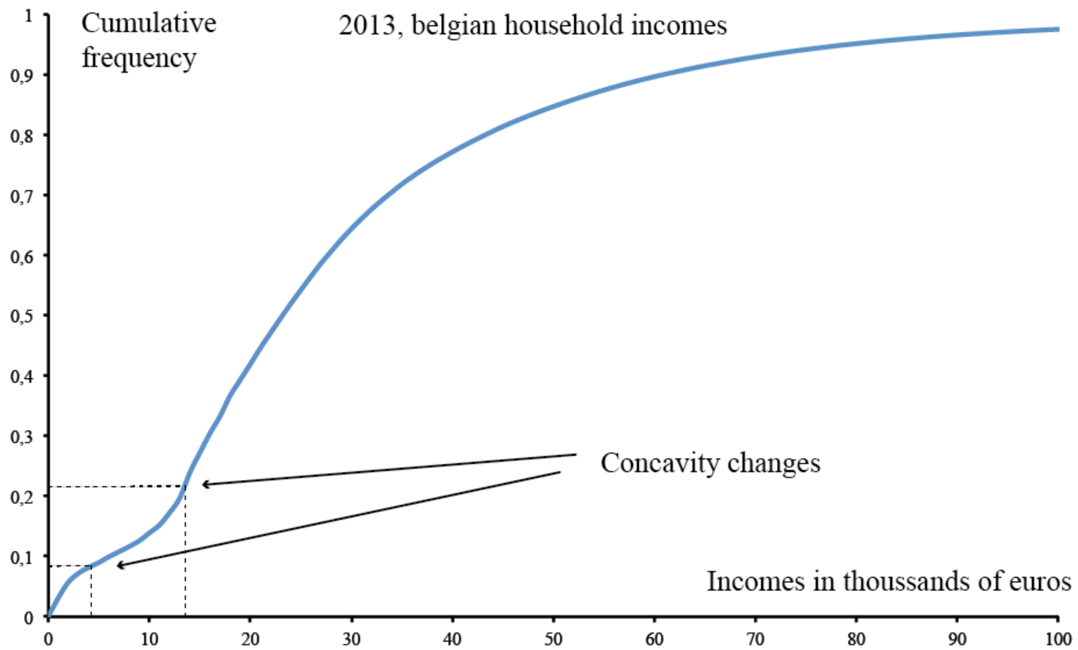



Fig. 35

The figure 35 gives the cumulative frequency for the belgian household incomes[4] in 2013. The cumulative frequency of a given value results from the sum of the frequencies associated with the values lower than the value of interest. When we add the maximal (or minimal) frequency of the

[4] The data is downloaded from the website of the belgian government : http://economie.fgov.be/fr/statistiques/chiffres/travailvie/fisc/

distribution, we obtain a maximum (of minimum) of growth of the cumulative frequency. The first derivative is maximal (or minimal). And the second derivative nullifies.

**Dias 36 and 37**

The appearance of a "bell curve" depends on the links between the mode, the mean and the median. If these 3 values are equal, it means that the curve is symmetric.

The figure 36 gives the scores obtained in mathematics by young pupils at the primary school term in "Fédération Wallonie-Bruxelles" (French speaking part of Belgium)[5]. The curve spreads toward the left and we see that the mode is higher than the median, which is higher than the mean.

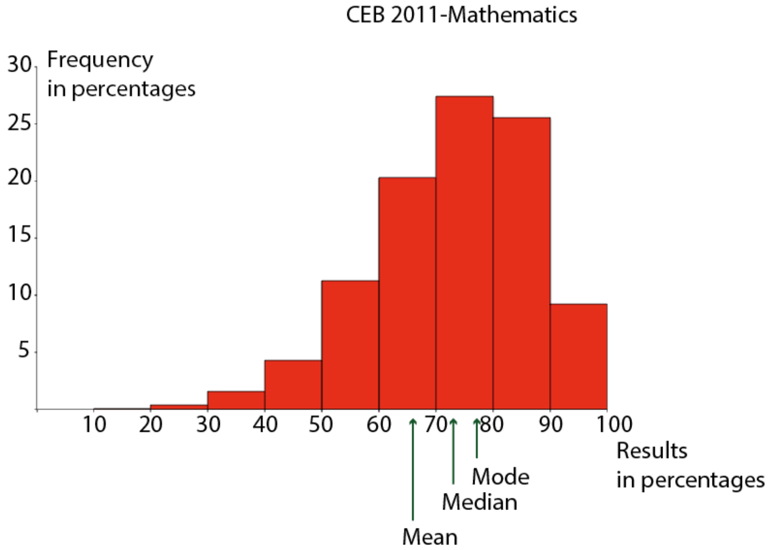



Fig. 36

The figure 37 shows the Belgian household incomes in the year 2013. This time, the curve spreads towards the right and the mode is lower than the median, which is lower than the mean.

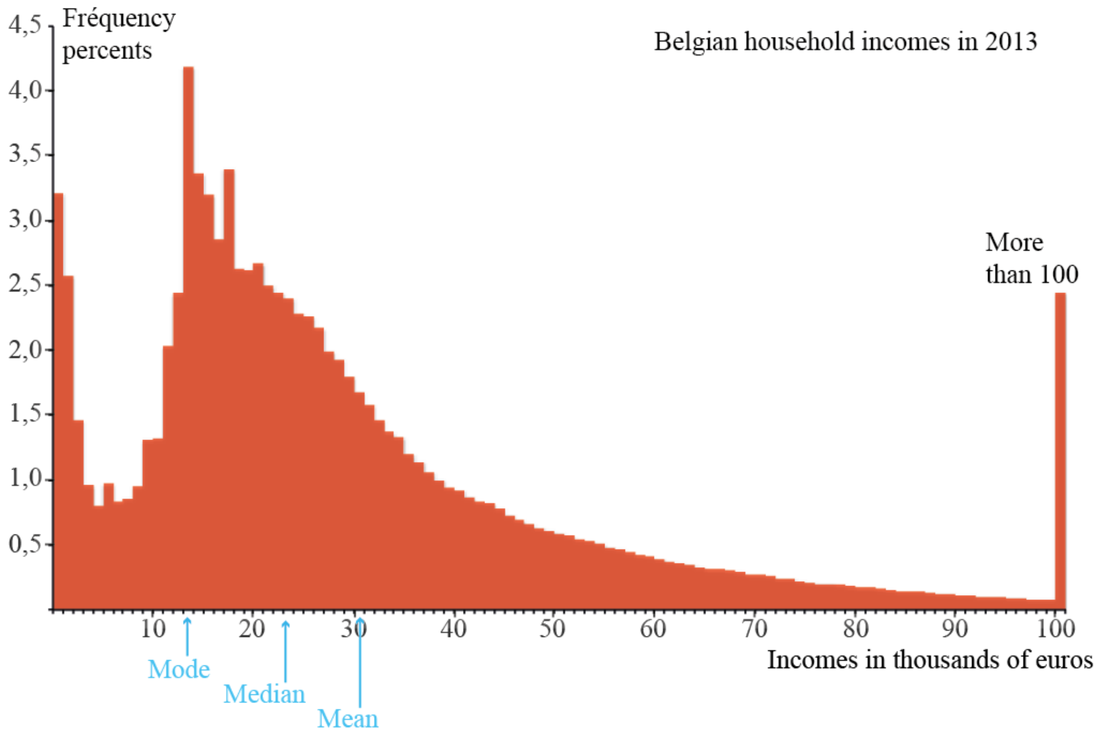



Fig. 37

**Dias 38 and 39**

Here is now the story of experiences concerning the basis of statistics, which were led in diverse classrooms. During the school year 2015-2016, we carried out our experiments in classes of :

- 5th and 6th years of primary school ;
- 1st year of secondary general school ;

[5] The data is downloaded from the website of the « Fédération Wallonnie-Bruxelles : http://www.enseignement.be/index.php?page=26754&navi=3376

- 3rd and 4th years of professional education ;
- 4th year of technical school.

We proposed an activity that included four workshops allowing to discover the main central values: mean, median and mode.

**Dia 40 and 41**

"1. Designation of a delegate

You have to choose a delegate to represent your class during class councils. Describe your method and your observations."

As developed earlier in the part covering the meanings and primary uses, the mode is more meaningful in a decision-making context, where the will of the majority is decisive. Based on this, we are more interested in the method than in the solution itself . We also wanted to initiate a debate on what seems right, fair, democratic, etc.

The first topic for the pupils was the selection criteria's leading to the choice of the "right" candidate i.e the one with adequate qualities to represent the class. Some suggestions were made: daily attendance, motivation, maturity, seriousness, sense of humour, capacity to generate idea, communication skills, beauty, capacity to be liked by the class, intelligence (or good grades), good team players, selflessness, listening capacity. Several pupils mentioned as well that a delegate should be "sapé comme jamais" (with reference to Mr. Gims song). Some pupils notice all the same the partiality of certain criteria's: " we could actually vote for the most beautiful while he/she is the most stupid, it does not make any sense! ". It is important to notice that in one of the classes, no qualities have been suggested because they considered these as "subjective" and consider that a delegate "should just have great ideas".

Beyond these attributes, other suggestions have recurred on a regular basis. First of all, delegates are elected after a vote. The pupils names are proposed, the votes are added up and the winner is the pupil who obtain the highest number of votes. He/she is therefore elected by the majority. Some pupils highlight the fact that the candidates have to be volunteer and the vote needs to be secret and personal (the persons involved have also to leave during the vote and the urn containing the ballot papers should be sealed). One pupil points out that this system is a democracy since everybody has the right to vote. Another one adds that a debate is necessary before voting in order for the candidates to be able to defend themselves, propose their ideas, defend their opinions, etc. and when the outcome is known, if somebody does not agree, the debate can start again and results into a second vote (idea of the electoral tour in France for instance).

Another method was mentioned in one of the classes: everybody writes the name of two persons whom he wants to see as delegate on a piece or paper, all the papers are placed in an urn and someone randomly picks one of them.

We have also observed the same idea in two different schools : the realization of a pie chart representing the preferences of everybody. Each pupil allocates individually the total i.e. 100 %, between their favourite candidates. Then, the graphs are pooled to come up with one for the class. The graphical representation had been covered beforehand in the class, which might have influenced the pupils.

In the primary school, several pupils suggest realizing posters the same way you would do for an election campaign. That could have an impact on the vote (who would vote for somebody doing a lot of spelling mistakes?). A new selection procedure has also been evoked: choose the delegate blindly i.e. pick up a delegate at random. The process is repeated "until we fall on a smart one". But then, how to agree on the level of intelligence of a person ? Furthermore, the action will be renewed until a consensus is reached. A new reaction was also mentioned : feeling envy or jealousy for the winner.

At the end of the focus group session, a brief synthesis was made in order to confront the ideas and to put appropriate words on the findings .

**Dias 42 and 43**

"2. “On prend son pied !”

What is shoe size of the pupil of the class who has as many peers whose shoe size is smaller than his, as peers whose shoe size is bigger ? Describe your method."

First of all, it is the formulation of this workshop which raised the more questions among us, for various reasons. On the one hand, we did not want to insert the expression " value of the middle " to avoid influencing them regarding the approach to follow. On the other hand, including the definition of median adapted to the situation within the problem statement was making it pretty complex to understand. Therefore, we were afraid that the difficulty of the workshop was linked to the assimilation of the instruction rather than in the resolution of the problem itself. The heaviness of the title also results from our willingness to be accurate : we are looking for a value (size) and not an individual (median pupil). It is the value dividing the series in two.

From a methodological point of view, we chose to collect the shoe size following the order of the benches i.e. school bench after school bench. All the results are noted on the board. To handle the odd cases and then the even cases, we add the professor shoe size in colour. It gives us two series of data, one with the shoe size of the professor and one without.

Regarding the choice of the context, we wanted to start from datas directly linked to pupils and easily available such as the size or the weight. However, these two might have created a certain embarrassment for some teenagers. The shoe size is less problematic.

As mentioned above, the formulation of the statement was complex and French is not necessary the mother tongue of pupils. Therefore, we decided to give a simpler and more concrete example (at the level of the mental representation) beforehand. Here it is: in a family of five children, which one has as many older than younger brothers and sisters ? The pupils understood rather fast that we were referring to the middle one. However, we asked them for more precision: how do you know that he is in the middle, how can you be sure ? It allowed us to highlight the necessity of classifying them, from the youngest to the oldest (or conversely, both solutions are correct), to determine which one is in the middle.

The pupils start by classifying the shoe sizes in the increasing or decreasing order (some make it at once, others take a bit more time). Then, they ask themselves: if we observe several times the same size, do we have to note all of them or is one time enough ? The opinions are shared... We discuss it to finally arrive to the conclusion that we have to take every observation into account, otherwise, the results are falsified. A pupil speaks then about including them twice and on her calculator, she writes 2×38. This brings the notion of the frequency of the values. A few pupils build a kind of table. They list all the values and write down the corresponding number of times they have been observed (frequency) for each of them, even for values not observed (in that case the number is zero).

Let’s consider the case where the number of sizes is odd. Several strategies emerge. Some pupils notice that 11 (for example) cannot be divided by two. Solution : we remove a size. Some groups start from both ends and move forward step by step until they have eliminated as many "pairs" of numbers as possible. Others do not notice the symmetry and make the counting from the left to the right : they count five sizes on the right, jump one (the middle one) and they check that there are five as well on the other side.

When the number of sizes is even, there is no value of the observable middle among the data. The pupils use the same principle as previously and quickly understand that there is an issue: there are two values in the middle ? Several proposals emerge : we take the biggest (or the smallest) value, we take the one who has the biggest (or the smallest) frequency (there is no agreement among the pupils). Some pupils suggest to pick a value randomly that could fit but the number of shoe sizes before and after do not match. Then, other pupils propose to choose the middle of these two values by adding them up and divide the result by 2 (which is the notion of mean even if this latter has not been covered yet). And this last measure is approved to go further.

In the primary school, when the number of sizes was even, we fell on a case where the two middle values were 38 and 38. A few considered that the value situated between 38 and 38 was 0. It can be explained by the "non-progress" on the line of numbers (to whom they refer regularly). For them, between 38 and 38, there is nothing and the value representing this absence is just “no value”. Others had proposed the value 19 How to explain it ? Our first assumption is that they chose the middle of 38 (that is between 0 and 38). The second is that the pupils, wanting to make the average of two consecutive numbers, may have integrated that it was enough to divide by two (in our case, 38 : 2 = 19. Another possibility is that they took into account only a half of the class at the time. If we look at the lowest half of the class and try to find out a value which is higher than 38, the children might only conceive it as “strictly higher” and therefore pick up 39. Yet, it is not possible given that what follows the first 38 is also 38. Then they try 38,5 because it looks acceptable. However, several of them quickly realise that it does not work.

**Dias 44 and 45**

" 3. For a good cause
a) In the context of the charity organised by the foundation Damien, 4 pupils have respectively bought 18, 12, 10 and 20 pens. Distribute them fairly among them. What is the part of every pupil, before and after the sharing ? Describe your method. "

This context allowed us to approach numerous themes of society such as the fair distribution of the properties and the sharing with people possessing less or nothing. Furthermore, the primary use of the average being an equitable sharing, it seemed relevant to us to use this method.

We chose the context of the pens of the operation Damien for various reasons : it allows the manipulation, it gives a meaning to the calculation of the mean (indeed, the sum of the number of pens has a sense, while a sum of numbers in the context of temperatures or sizes does not), it speaks to pupils who know this charity, and it does not foster envy or egoism like it could in a context of money. Furthermore, money would not have allowed us to address the characteristics of the mean, for instance, the fact it can be a non-observable value. Indeed, obtaining 12,4 euros does not raise any question but it does when talking about 12,4 pens…

To distribute fairly the pens to each pupils, several strategies were considered.

- We count the total number of pens and we divide it by the number of persons present (4 in this case). Many groups used this formula immediately, without using the material. Is it the result of an intuition, an habit, a reflex ?
- We gather all the pens in a pile and we distribute them as a game of cards, one by one. We observe that a few pupils react : as each of them has already at least 10 pens, everybody keeps his 10 pens and we share the remaining amount, which is called fair sharing.
- In other groups, the pupils (who know the total number of pens after the sharing 60: 4 = 15) are going to pick some in the closest pile to complete their collection. We observe here the method of compensation (we take what we are lacking to those who have too much). However, they use this process without any logic. Indeed, they do not necessarily go towards the pupils who have too many pens, but towards a pile chosen at random. The figure 45 show the solution of a group who proposes a clear and detailed compensation.

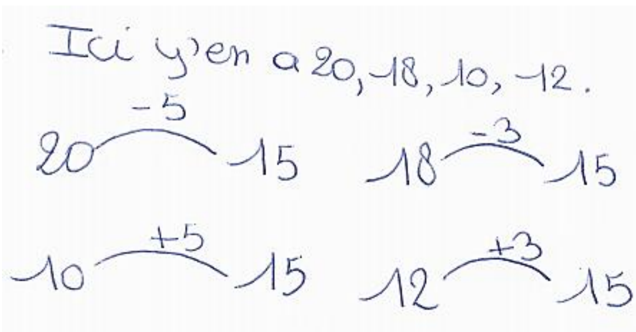



Fig. 45

After manipulation, we try to find a calculation allowing us to illustrate the method used (for those who had not evoked it yet). The pupils can therefore confirm that the manipulation was correctly performed.

An interesting method also drew our attention : represent by a fraction the proportion of pens that every pupil has before and after the sharing.

Generally speaking, we noticed that this activity is the one which raised the less questions and issues and seemed the most logical and obvious for the pupils.

**Dias 46 and 47**

"3. For a good cause

b) Jean-Luc, who didn't have the opportunity to buy pens for a good cause, joins the group. What is then the new part of each pupil ? Describe your method and your observations."

We approach the notion of adding a null value that is to be considered in the calculation of a mean. The sense of solidarity and sharing will be severely strained, so that we expect likely oppositions.

Again, several strategies emerge…

- We adapt the first method to this exercise: we sum the number of pens (which remains unchanged since Jean-Luc arrives empty-handed) and divide the result by the number of people that now amounts to five.
- We set again a threshold of 10 pens per person and fairly distribute the rest (one by one). If it turns out that we are lacking pens, the pupils already think of fixing a lower threshold, 5, for example.
- Many groups, knowing the method, do the math, reach the conclusion that each person should receive 12 pens, and then adapt their pile (we observe the opposite situation compared with the part a), in which most of the groups had begun by manipulating the pens).
- We also observed a very practical method that we had not thought of before : they started from the previous piles (15 pens) and, knowing that each pupil should receive 12 pens, they took 3 pens from every pile to form a new one (figure 47).

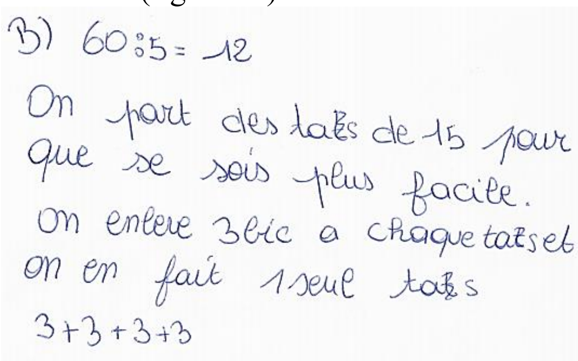



Fig. 47

- Starting from the sharing done for the case a), a pupil suggested that each member of his group (they were four) gives a pen to Jean-Luc (played by one of us). Each of them, having 15 pens at first, gave one to Jean-Luc, then a second… But they decided to stop this distribution, considering that Jean-Luc would receive far too many pens if they were continuing. If they put an end to the process, it is because they though in terms of fair sharing amongst them four and Jean-Luc and not amongst all the five. They anticipated a process that, after a number of steps, would have led them to a completely unfair distribution, whereas while carrying out the operation, they could have noticed that the objective was reached after only three steps.
- Some pupils considered the case in which the number of pens would not be divisible by 5 (it is actually the objective of the part c).

To our surprise, when we asked the pupils if the arrival of Jean-Luc, who had its hands empty, did not bother them, their answer was negative. Indeed, a null value must be taken into account even if it reduces the mean. Unanimously, they answered that they did not mind, that on the contrary it was necessary to have the team spirit and to show solidarity. Let us highlight that in one class (secondary school), the pupils had not the same opinion and did not accept the coming of Jean-Luc so easily.

The pupils observed the effect of the null value on the mean, it pulled the latter downwards and each participant is forced to decrease his theoretical number of pens.

**Dia 48 and 49**

" 3. For a good cause
c) In another group, 5 pupils bought respectively 16, 12, 8, 20 and 8 pens. Calculate the average number of pens bought by these pupils. "

This new situation is similar to the first one except for the fact that the average number of pens is not an integer and is thus not a physically observable value. During the first experiment, points b) and c) were grouped in a single one. It means that the particularities of adding a null value and of an unobservable average were encountered in the same question. Yet, these are two steps difficult to overcome and we thus though pertinent to address them separately.

For this last part, the pupils did not have pens at their disposal anymore. Automatically, all the groups summed the pens and divided the total amount by the number of persons present.

Once the calculation is done, here is the result: 12,8! “Oh, I made a calculation error!”. “No, it’s correct”.

Several proposals then emerged…

- Remove pens so that the calculation results in a whole number (we round off to the lower unit) : no, all the pens must be taken into account.
- Add pens for the same reason (we round up to the superior unit) : nor is it a solution as it distorts the results.
- Remove or add people : no, once again, it distorts the results.
- Give the remaining pens to the “poor” people.
- Some people will possess 12 pens and some others 13. The pupils think that it is not possible to achieve a fair sharing.
- Last solution, rather original : the professor recovers the surplus of pens and resells them.

The pupils being sceptics, we provided them a more telling example. We asked to each of them to give us the number of children in his family. We then calculated the mean number of children by family. We obtained, for example, 2,8. Hence the question : does it mean that there are 2,8 children in each of your families ? No, it is the value that gives the theoretical number of children if we had wanted to distribute them in a fair way between all the families. We therefore conclude that a mean is not necessarily an observable value, a value included in the dataset.

**Dias 50 to 53**

" 4. What value to choose?
During a dinner, a group of 10 friends played a tournament of darts. The table 50 give the results. What is the most representative result of the game level of the group of friends ? Justify your choice by describing your method and observations. "

Tab 50

| Prénom | Score |
|---|---|
| Marie | 44 |
| Arnaud | 43 |
| Caroline | 37 |
| Dylan | 41 |
| Stéphane | 41 |
| Romane | 251 |
| Fanny | 39 |
| Maurine | 37 |
| François | 49 |
| Laura | 45 |

To close the discovery of the representative values, we felt it would be interesting to combine them within a new situation. This last workshop requires the use of three notions previously presented (mode, median and mean), their comparison and the selection of the one that will be the most representative of the given situation.

Too often, the pupils tend to use the mean indiscriminately, without being able to provide a justification for this use. We chose to look at extreme values (251, score of Romane), in particular to show that the mean is not a robust value.

Furthermore, we decided to confront the learners with a multimodal series (indeed, 37 and 41 appear twice). It is not the same situation as in the first activity : the equality was resolved with a second round of elections.

We chose to present the raw data (in the form of a table) asking for a classification. At the end of the workshop, we plan to launch a debate on the validity of these different values in the present situation and on whether the presence of Romane is accepted or not. Other examples of changes affecting certain scores will be proposed in the groups of pupils so that we can indentify the potential ongoing difficulties (example: if Romane had not obtained 250, but 0, how would vary the values).

For the pupils, a representative value is a value that can be observed. They thus look a bit randomly for a value that seems appropriate within the table of data. They notice that most of the people have between 30 and 50 points, apart from Romane (extreme value). The presence of this latter does not shock anybody before we point it out. Some people want to divide the Roman's score by two or more to obtain several more consistent scores. The pupils notice very quickly that the scores 37 and two 41 appear twice. For some pupils, the most representative score of the classroom is the one of Romane, because it is the highest. We realize that the pupils understand the notion of "representativeness" in several different ways.

We then address the different central values, not necessarily in the order of the previous workshops (it varies from group to group).

In this series of data, we observe two modes : so what can be done ? Several suggestions are made : select the smallest, the largest, the mean of the modes, or take the value 41 because 37 is too small (subjective reason). A pupil also suggests performing this calculation:

$$(2 \times 41 + 2 \times 37)/4$$

Which amounts to calculate a kind of mean of the modes while accounting for their frequencies. Another one points out that the mode cannot be representative of the series because it does not take into account all the values (very relevant intervention!).

Concerning the mean, the pupils immediately think of adding all the scores and dividing the sum by the number of people. No one is surprised by the particularly high average  (62,7), which results from the presence of the extreme value (Romane) pulling the average upwards (except for some pupils who do not consider it as representative of the data series and believe that it is thus necessary to find another method). As for the presence of Jean-Luc in the activity involving pens, very few young people want to exclude Romane, perceived as a cheater given her very high score. To illustrate the situation, a pupil very rightly said: " It is like Usain Bolt, he runs much faster than the others, but he is a member of the championship! We have to take him into account ". In some groups, they calculate the average with and without Romane to see the difference. Finally, the pupils realize how much an extreme value may influence the mean.

An unexpected method was also proposed, which consists of taking the least good and the best scores and then compute their average, what means calculating the midrange.

For the median, we notice that many groups (about one out of two) search for the middle value while forgetting to order the data. With and without Romane, the median only changes by a single unit. it is thus a more reliable central value. Some people also prefer this value because, in their opinion, it is faster to determine.

After discussion, we reach the conclusion that, in the case in point, the median seems to be a more representative value than the mean, as the first one is closer to the majority of the scores.

**Dia 54 à 56**

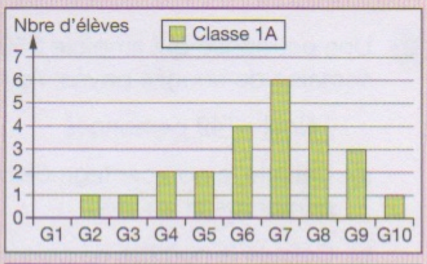
Limits of the central values

The figure 54 comes from a school book (first year of secondary school) : "Ph Ancia, M. Bams, M. Colin, P. Dewaele, F. Huin, A. Want, Actimath à l'infini 1, Cahier d'activités, Van In, Wavre-Wommelgem, 2013. Let's look at the question e) : "What is the best class ?"

Les diagrammes ci-dessous présentent les résultats obtenus à Noël au cours de mathématique par les élèves de deux classes. Ces élèves sont répartis en 10 groupes selon leur pourcentage.

G1 : de 0 à moins de 10 %  G2 : de 10 à moins de 20 %
G3 : de 20 à moins de 30 %  G4 : de 30 à moins de 40 %
G5 : de 40 à moins de 50 %  G6 : de 50 à moins de 60 %
G7 : de 60 à moins de 70 %  G8 : de 70 à moins de 80 %
G9 : de 80 à moins de 90 %  G10 : de 90 à 100 %

a) Détermine, pour chaque classe, le nombre d'élèves de chaque groupe.
b) Détermine le pourcentage des élèves de chaque classe qui ont un échec.
c) Détermine le pourcentage des élèves de chaque classe qui ont au moins 50 %.
d) Détermine le pourcentage des élèves de chaque classe qui ont plus de 80 %.
e) Avec un tel graphique, peux-tu dire quelle est la classe qui a le meilleur résultat ?



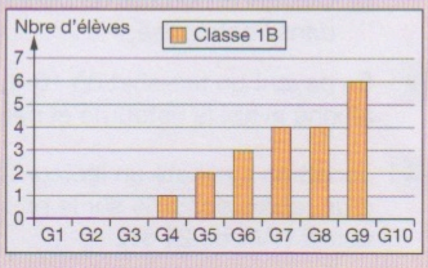




Fig. 54

1. At the basic level
Based on the graph, it may seem that the class B is characterized by higher values…
Let's see the students' answers (high school[6] to become mathematics teacher).

The class 1B is the best one, because it is characterized by the lowest failure rate and the highest percentage of scores above 80 %.

What's the meaning of « better than » ? The largest mean ? The lowest failure rate ? The highest maximum ?

The class 1B seems to achieve better scores because the dispersion is lower and the values are around G7 and G8.

2. From a broader perspective
Let's construct the statistical table (table 55.1) and compute the central values (table 55.2).

[6] In the "Fédération Wallonie-Bruxelles", the high school follow the secondary school and correspond to bachelor program.These students are about 21 years old.

| From | To | Middle | Class A Absolute frequency | Class A Relative frequency | Class A Cumulative frequency | Class B Absolute frequency | Class B Relative frequency | Class B Cumulative frequency |
|---|---|---|---|---|---|---|---|---|
| | 0 | | | | 0 | 0 | 0 | 0 |
| 0 | 10 | 5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 20 | 15 | 1 | 0,042 | 0,042 | 0 | 0 | 0 |
| 20 | 30 | 25 | 1 | 0,042 | 0,083 | 0 | 0 | 0 |
| 30 | 40 | 35 | 2 | 0,083 | 0,167 | 1 | 0,05 | 0,05 |
| 40 | 50 | 45 | 2 | 0,083 | 0,250 | 2 | 0,1 | 0,15 |
| 50 | 60 | 55 | 4 | 0,167 | 0,417 | 3 | 0,15 | 0,3 |
| 60 | 70 | 65 | 6 | 0,250 | 0,667 | 4 | 0,2 | 0,5 |
| 70 | 80 | 75 | 4 | 0,167 | 0,833 | 4 | 0,2 | 0,7 |
| 80 | 90 | 85 | 3 | 0,125 | 0,958 | 6 | 0,3 | 1 |
| 90 | 100 | 95 | 1 | 0,042 | 1,000 | 0 | 0 | 1 |
| Total number | | | 24 | | | 20 | | |

Tab. 55.1

| | Classe A | Classe B |
|---|---|---|
| moyenne | 60,83 | 68 |
| médiane | 63,3 | 80 |
| mode | 65 | 85 |

Tab. 55.2

Let’s have a look at the graph of cumulative frequencies (figure 56). Until 90 %, the curve corresponding to the class 1B is lower than the curve of the class 1A. The class B is the best, isn’t it ? There are less scores below 40 %, 50 %, 60%...

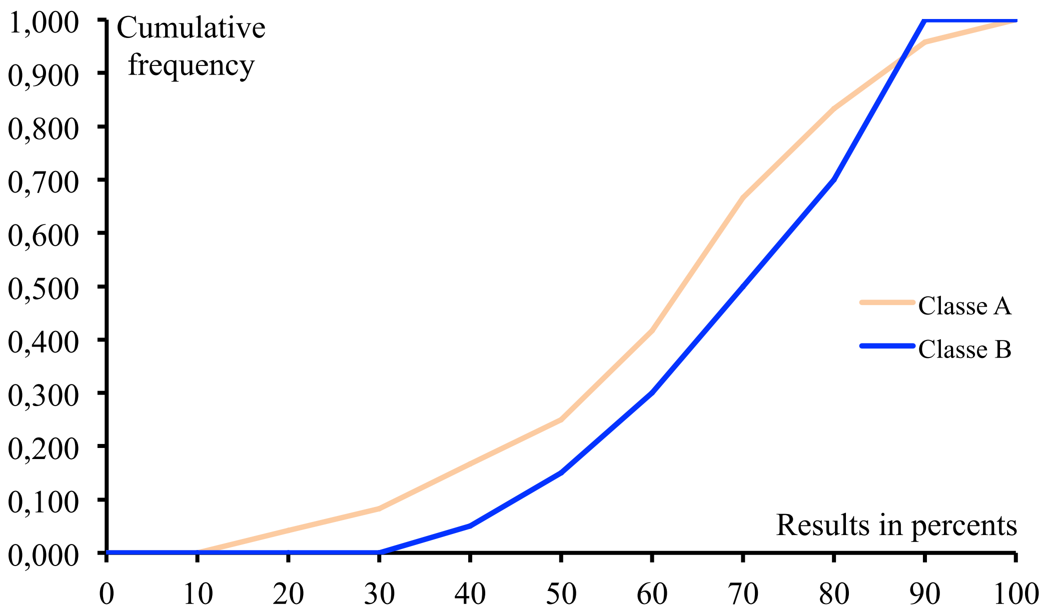



Fig. 56

3. From an even broader

Let us test the difference in the means of the two classes to see if it is significant. We consider two random variables associated with each of the classes: $X_A$ et $X_B$ . The following hypotheses are tested using a significance level $\alpha = 0,05$ :

$$H_0 : \mu_A = \mu_B$$

$$H_1 : \mu_A \neq \mu_B$$

$X_A$ et $X_B$ don't follow a normal distribution, the standard deviations are not known and are supposed to be unequal. We thus consider the Student t-test.
From the statistical table, we can compute the means and the standard deviations of the samples:

$$n_A = 24 \quad n_B = 20 \quad \mu_A \cong 60,83 \quad \mu_B = 68 \quad s_A^2 \cong 399,275 \quad s_B^2 \cong 243,158 .$$

Thus

$$\sigma_{\bar{X}_B - \bar{X}_A} = \sqrt{\frac{s_A^2}{n_A} + \frac{s_B^2}{n_B}} \cong 5,366$$

The number of degrees of freedom is

$$v = \frac{\left(\frac{s_A^2}{n_A} + \frac{s_B^2}{n_B}\right)^2}{\frac{1}{n_A+1}\left(\frac{s_A^2}{n_A}\right)^2 + \frac{1}{n_B+1}\left(\frac{s_B^2}{n_B}\right)^2} - 2 \cong 44$$

We have to verify that

$$\mu_B - \mu_A \in \left[0 - t_{0,025;44}.5,366;\ 0 + t_{0,025;44}.5,366\right]$$

Or

$$7,17 \in \left[-2,0154.5,366;2,0154.5,366\right] \cong \left[-10,8;10,8\right]$$

It's the case, $H_0$ is accepted. We cannot assert that there is a statistically significant difference between the classes. How to explain that the first intuition was not the good one ? It is due to the relatively important standard deviations. For the first class, this deviation is equal to 19,98 and for second, to 15,59. The difference in the averages is lower than these two values, what means that the dispersal of the scores in each of the classes is larger than the difference between the two classes …

The central values are powerful tools and allow to handle numerous problems, but they have their limits as we've seen with this last example.